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EXACT ANALYSIS OF BEAMS ON TWO-PARAMETER ELASTIC FOUNDATIONS

A. GHANI RAZAQPUR and K. R. SHAH
Department of Civil Engineering, Carleton University, Ottawa, Canada K1S 5B6

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Abstract—Efficient beams on two-parameter elastic foundation finite elements have recently been developed. The stiffness matrix and nodal load vector of these elements have been derived on the basis of the exact displacement function obtained from the solution of the governing differential equation. Most of the existing elements are, however, either limited to certain combinations of beam and foundation parameters, or provide only the solution of the homogeneous form of the governing equation. In this paper a new finite element is derived which eliminates these limitations. The stiffness matrix, nodal load vector and shape function of the element are derived using the differential equation of a beam on a two-parameter elastic foundation. The complete solution of the equation corresponding to the most common types of load is also presented. This permits the determination of the deflections and internal forces anywhere along a simple or continuous beam on two-parameter foundations.

NOTATION

The following symbols are used in this paper:

a	defined by eqn (29f)
A_1–A_4	constants of integration
$\{A\}$	matrix containing A_1–A_4
b	defined by eqn (29e)
B	width of beam
c	defined by eqn (29h)
c_1–c_4	constants of integration
d	defined by eqn (29g)
D_1, D_3	beam element end displacement
D_2, D_4	beam element end rotation
$\{D\}$	matrix containing degrees of freedom D_1–D_4
e	base of natural logarithm
E	beam element elastic modulus
$\{E\}$	matrix relating displacements $\{D\}$ to constants $\{A\}$
E_s	Young's modulus of foundation
$\{F\}$	matrix of beam end forces and moments
$\{G\}$	inverse of matrix $\{E\}$
$\{H\}$	matrix relating constants $\{A\}$ to end forces $\{F\}$
I	beam element moment of inertia
k	first parameter of elastic foundation or Winkler foundation modulus
k_1	second parameter of elastic foundation
k_{ij}	elements of stiffness matrix
k_G	a parameter of the shear layer
k_θ	reaction moment per unit length per unit rotation
L	beam element length
m, n	coefficients of linear function defining the particular solution of eqn (1)
$m(x)$	applied distributed moment
$M(x)$	bending moment at a section a distance x from the beam end
M_0	applied concentrated moment
$\{N\}$	matrix of exact shape functions of beam element
$\{P\}$	matrix of equivalent nodal loads
$p(x)$	elastic foundation pressure
P_0	applied concentrated load
$q(x)$	applied distributed load
Q	axial load
r	$\cosh \alpha L$
R	$\cosh \beta L$
R_1	$\sin \beta_1 L$
$[S]$	beam element stiffness matrix
t	$\sinh \alpha L$
T	$\sinh \beta L$
T_m	constant tension in an elastic membrane connecting top ends of springs

T_1	$\sinh \beta_1 L$
u, v	coefficients of linear function defining $q(x)$
$V(x)$	generalized vertical shear at a section a distance x from the beam end
$V_n(x)$	generalized normal shear at a section a distance x from the beam end
w	beam lateral displacement
W	defined by eqn (29a)
$w_{f\,end}$	free end vertical displacement
w_h	solution of homogeneous form of eqn (1)
w_p	particular solution of eqn (1)
w_1	beam displacement due to nodal displacements and rotations
w_2	displacement due to applied loads in a fixed-end beam
x	distance along the beam axis
x_1	distance from the beam end to where $q(x)$ begins
x_2	distance from the beam end to where $q(x)$ ends
X	defined by eqn (29b)
Y	defined by eqn (29c)
Z	defined by eqn (29d)
α	$\sqrt{\dfrac{k_1}{4EI}+\sqrt{\dfrac{k}{4EI}}}$
β	$\sqrt{\dfrac{k_1}{4EI}-\sqrt{\dfrac{k}{4EI}}}$
β_1	$\sqrt{\sqrt{\dfrac{k}{4EI}}-\dfrac{k_1}{4EI}}$
γ	$\alpha-\beta$
Δ	$\dfrac{\beta^2 t^2-\alpha^2 T^2}{\beta}$
Δ_1	$\left(\dfrac{\beta_1^2 t^2-\alpha^2 T_1^2}{\beta_1}\right)$
ν_s	Poisson's ratio of foundation
μ	constant expressing rate at which vertical deformation of foundation decays with depth
λ	$\alpha+\beta$
λ_1	$\alpha^2+\beta^2$
λ_2	$2\alpha\beta$
λ_3	$\beta^3+3\alpha^2\beta$
λ_4	$\alpha^3+3\alpha\beta^2$
λ_5	$\alpha^2-\beta_1^2$
λ_6	$2\alpha\beta_1$
λ_7	$-\beta_1^3+3\alpha^2\beta_1$
λ_8	$\alpha^3-3\alpha\beta_1^2$
If	$k_1>\sqrt{4kEI}$
ϕ_1	$\cosh \alpha x \cosh \beta x$
ϕ_2	$\cosh \alpha x \sinh \beta x$
ϕ_3	$\sinh \alpha x \cosh \beta x$
ϕ_4	$\sinh \alpha x \sinh \beta x$
If	$k_1<\sqrt{4kEI}$
ϕ_1	$\cosh \alpha x \cos \beta_1 x$
ϕ_2	$\cosh \alpha x \sin \beta_1 x$
ϕ_3	$\sinh \alpha x \cos \beta_1 x$
ϕ_4	$\sinh \alpha x \sin \beta_1 x$.

INTRODUCTION

Recently, beams on two-parameter elastic foundation have received considerable attention (Zhaohua and Cook, 1986; Eisenberger and Clastornik, 1987; Chiwanga and Valsangkar, 1988; Valsangkar and Pradhanang, 1988; Karmanlidis and Prakash, 1989). Zhaohua and Cook (1986) discussed the different types of elastic foundation models and developed the stiffness matrix and nodal load vector of a beam on a two-parameter elastic foundation finite element. The two-parameter foundations that this element can model include the Filonenko–Borodich, Pasternak, Generalized, Vlasov and Winkler models. They derived the stiffness matrix of the element in two ways: first, based on the cubic displacement function used for ordinary beams without elastic foundation; secondly, based on the exact displacement function obtained from the solution of the differential equation governing the behavior of beams on two-parameter foundations. They concluded that while in some cases 80 elements of the type based on the cubic function may be needed to obtain the converged solution of the problem, the same problem can be solved using one or two elements of the kind based on the exact displacement function. Hence, the use of the latter element leads to a noticeable amount of saving in computer resources and human effort when solving problems of beams on two-parameter elastic foundations.

Unfortunately, their solution was limited to certain combinations of beam and foundation stiffnesses. Chiwanga and Valsangkar (1988) developed a beam element on a two-parameter elastic foundation and gave its nodal load vector, but their solution was also limited by the same constraint. Eisenberger and Clastornik (1987) solved the problem of a beam on a variable two-parameter elastic foundation and obtained the solution of the governing differential equation by means of an infinite polynomial series. Their solution is again primarily concerned with the homogenous form of the equation. The accuracy of this solution naturally depends upon the number of terms used in the series. Although some guidelines are provided with respect to the choice of this number (the authors used 80–110 terms to obtain the converged solution for a cantilever beam loaded at its tip), the required number of terms will obviously be dependent upon the beam loading and boundary conditions. Furthermore, in the case of constant foundation parameters, a non-series exact solution is simpler because it will have only four trigonometric or hyperbolic terms for the homogeneous part plus two to three terms for the particular solution corresponding to most common types of loads. Valsangkar and Pradhanang (1988) provided the well-known solution for the homogeneous form of the differential equation governing the dynamic behaviour of beam-columns on a two-parameter foundation for different foundation parameters and stiffness combinations. They did not, however, present any stiffness matrices or the complete solution of the problem for the nonhomogeneous form of the equation corresponding to the usual load combinations. Karmanlidis and Prakash (1989) gave transfer and stiffness matrices for all the possible cases of beam-columns on a two-parameter elastic foundation but did not discuss the procedures for obtaining the equivalent joint loads, the particular solution of the problem, or the methods for calculating the final displacements, shearing forces and bending moments. These procedures are necessary to obtain a complete solution of the problem.

In this paper, a beam element is presented which permits the solution of beams on any type of constant two-parameter elastic foundation. The explicit form, completeness and simplicity of the present solution are its principal advantages. The stiffness matrix, shape functions and nodal load vector corresponding to the most important types of loads are given in explicit form. The complete solution of the governing differential equation is presented in a form which can easily be implemented in the ordinary frame analysis computer programs based on the stiffness method. The authors believe that provision of stiffness matrices alone, as in most of the references cited earlier, often does not provide the necessary information in a form which would permit the user to exploit, without extensive effort, the advantages of an exact displacement function versus a solution based on cubic polynomial shape functions. The reason for this is that exact stiffness matrices should be accompanied by the exact nodal load vector and the exact particular solution corresponding to practical loading cases in order to realize the above advantages. Finally, it should be stated that this paper is not intended to cover the problem of beam-columns on elastic foundations, as presented by Karmanlidis and Prakash (1989), although the procedure presented here can be extended, in conjunction with the solution presented by the above authors, to deal with the problem of beam-columns on an elastic foundation.

GOVERNING DIFFERENTIAL EQUATION

The equation governing the behavior of beams on a two-parameter elastic foundation is given by

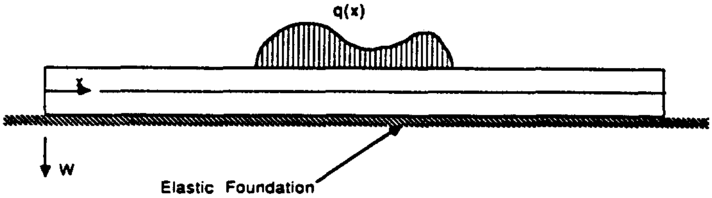

Fig. 1. Typical beam on two-parameter elastic foundation.

$$EI\frac{d^4w}{dx^4} - k_1\frac{d^2w}{dx^2} + kw = q(x) \tag{1}$$

where w is the displacement function, EI is the beam flexural rigidity, $q(x)$ is the applied loading function, k is the first foundation parameter, usually referred to as the Winkler foundation modulus, and k_1 is the second foundation parameter which has different definitions, depending on the two-parameter foundation model being utilized; see Fig. 1. For the most common types of foundation models, k_1 is given as follows.

Filonenko–Borodich foundation

$$k_1 = T_m \tag{2a}$$

where T_m = constant tension in an elastic membrane connecting the top ends of Winkler-type springs.

Pasternak foundation

$$k_1 = k_G \tag{2b}$$

where k_G = a parameter of the shear layer.

This model is based on the assumption that there is shear interaction between the springs, and the top ends of the springs are connected to an incompressible layer which resists only transverse shear deformations.

Generalized foundation

$$k_1 = k_\theta \tag{2c}$$

where k_θ = reaction moment per unit length per unit rotation.

This model assumes that at the point of contact between beam and foundation there is not only pressure but also moments. These moments are assumed to be proportional to the angle of rotation and the second parameter is the constant of proportionality.

Vlasov foundation

Here the foundation is treated as a semi-infinite medium and simplifying assumptions are made to obtain the second parameter in terms of elastic constants and the dimensions of the beam and the foundation:

$$k_1 = \frac{E_s}{4(1+\nu_s)}\frac{B}{\mu} \tag{2d}$$

where

E_s = Young's modulus of foundation
ν_s = Poisson's ratio of foundation
B = width of beam

μ = constant expressing the rate at which vertical deformation of foundation decays with depth (Scott, 1981).

k and k_1 can be used to calculate the foundation pressure, $p(x)$,

$$p(x) = kw - k_1 \frac{d^2w}{dx^2}. \tag{3}$$

The solution of eqn (1) may be written as

$$w = w_h + w_p \tag{4}$$

where w_h is the solution of the homogeneous form of the equation and w_p is a particular integral corresponding to $q(x)$. When dealing with the derivation of the stiffness matrix, we need only consider the homogeneous solution

$$w = w_h = A_1\phi_1 + A_2\phi_2 + A_3\phi_3 + A_4\phi_4 \tag{5}$$

where A_1–A_4 are constants, and ϕ_1–ϕ_4 are four linearly independent functions whose exact form depends on the relative magnitude of EI, k and k_1. Since the latter quantities are rigidity parameters of beam and foundation, they are all non-negative. Thus, there are only three possible combinations of the parameters that need to be considered, i.e. k_1 larger than, equal to or smaller than $\sqrt{4kEI}$. For $k_1 > \sqrt{4kEI}$, ϕ_1–ϕ_4 are listed below:

$$\phi_1 = \cosh \alpha x \cosh \beta x \tag{6a}$$

$$\phi_2 = \cosh \alpha x \sinh \beta x \tag{6b}$$

$$\phi_3 = \sinh \alpha x \cosh \beta x \tag{6c}$$

$$\phi_4 = \sinh \alpha x \sinh \beta x \tag{6d}$$

where α and β are as follows:

$$\alpha = \sqrt{\frac{k_1}{4EI} + \sqrt{\frac{k}{4EI}}} \tag{7a}$$

$$\beta = \sqrt{\frac{k_1}{4EI} - \sqrt{\frac{k}{4EI}}}. \tag{7b}$$

Generally, $k_1 < \sqrt{4kEI}$ is satisfied by most physical problems, as has been noted by others (Scott, 1981; Zhaohua and Cook, 1986). Therefore, most of the existing beam on two-parameter foundation finite elements deal with this case only but cannot be applied to $k_1 > \sqrt{4kEI}$. However, when performing nonlinear analysis, in which the flexural rigidity of the beam and the soil parameters can vary widely, depending on the level of stress, the full solution of the differential equation would be needed to solve the problem. The objective of this paper is to develop an element that would eliminate the difficulties that may arise due to $k_1 > \sqrt{4kEI}$.

When $k_1 < \sqrt{4kEI}$, ϕ_1–ϕ_4 are as follows:

$$\phi_1 = \cosh \alpha x \cos \beta_1 x \tag{8a}$$

$$\phi_2 = \cosh \alpha x \sin \beta_1 x \tag{8b}$$

$$\phi_3 = \sinh \alpha x \cos \beta_1 x \tag{8c}$$

$$\phi_4 = \sinh \alpha x \sin \beta_1 x \tag{8d}$$

where β_1 is given by

$$\beta_1 = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{k_1}{4EI}}. \tag{9}$$

It is possible to get the exact solution for $k_1 = \sqrt{4kEI}$, but in practice accurate results can be obtained by just increasing k_1 by a very small amount and then using the solution for $k_1 > \sqrt{4kEI}$. For this reason, the exact solution for $k_1 = \sqrt{4kEI}$ is not presented here. In the ensuing developments, the formulation for $k_1 > \sqrt{4kEI}$ is presented in detail. The formulation for $k_1 < \sqrt{4kEI}$ can be similarly performed, and is given in Appendix B for the sake of completeness.

For convenience, eqn (5) may be written in matrix notation

$$w = \{\phi\}^{\mathrm{T}}\{A\} \tag{10}$$

where matrices $\{\phi\}$ and $\{A\}$ contain ϕ_1–ϕ_4 and A_1–A_4, respectively, and the superscript T denotes the transpose of a matrix. We shall use eqn (10) to develop the exact shape functions, stiffness matrix and nodal load vector of a beam on a two-parameter elastic foundation finite element.

EXACT SHAPE FUNCTIONS

Consider the beam element in Fig. 2, which has a length L and four degrees of freedom — D_1 to D_4 — at the two ends or nodes. Note that D_1 and D_3 are end displacements while D_2 and D_4 are end rotations. Associated with the generalized nodal displacements, $\{D\}$, are the generalized nodal forces, $\{F\}$, which consist of shearing forces and bending moments. $\{D\}$ can be written as

$$\{D\}^{\mathrm{T}} = \{(w)_{x=0}, (w')_{x=0}, (w)_{x=L}, (w')_{x=L}\} \tag{11}$$

where the prime denotes the derivative of w with respect to x.

Substituting for w and its derivative from eqn (10) into eqn (11) results in:

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \alpha & 0 \\ Rr & Tr & Rt & Tt \\ (\beta Tr + \alpha Rt) & (\beta Rr + \alpha Tt) & (\beta Tt + \alpha Rr) & (\beta Rt + \alpha Tr) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{12}$$

or

$$\{D\} = [E]\{A\} \tag{13}$$

where $r = \cosh \alpha L$, $R = \cosh \beta L$, $t = \sinh \alpha L$, $T = \sinh \beta L$ and $[E]$ is the 4×4 matrix in

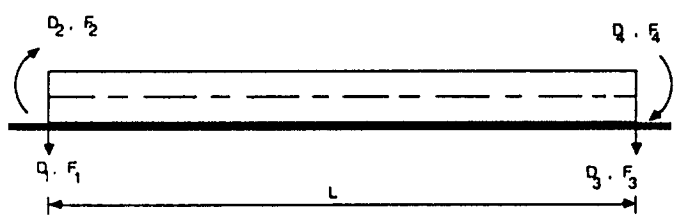

Fig. 2. Element end displacements and forces.

eqn (12). It should be mentioned that when T is used as superscript it means transpose, otherwise it always denotes sinh βL. Equation (13) can be solved for the unknown constants $\{A\}$

$$\{A\} = [E]^{-1}\{D\}$$

or

$$\{A\} = [G]\{D\}. \tag{14}$$

The matrix $[G] = [E]^{-1}$ is explicitly given in Appendix A. Substituting for $\{A\}$ from eqn (14) into eqn (10) yields

$$w = \{\phi\}^{T}[G]\{D\}, \tag{15}$$

or

$$w = \{N\}^{T}\{D\} \tag{16}$$

where

$$\{N\}^{T} = \{\phi\}^{T}[G]. \tag{17}$$

The vector $\{N\}$ is a 4×1 matrix whose elements are the exact shape functions of the beam on a two-parameter elastic foundation finite element. It may be recalled that each shape function describes the equation of the elastic curve when the beam is given a unit displacement in the direction of one of the degrees of freedom while the remaining degrees of freedom are set equal to zero.

A typical shape function corresponding to $D_1 = 1$, $D_2 = D_3 = D_4 = 0$, is given by

$$N_1 = G_{11}\cosh\alpha x\cosh\beta x + G_{21}\cosh\alpha x\sinh\beta x + G_{31}\sinh\alpha x\cosh\beta x + G_{41}\sinh\alpha x\sinh\beta x \tag{18}$$

where the G_{ij} are elements of $[G]$. This is in contrast to the corresponding shape function of a conventional beam, i.e. a beam without elastic foundation, which is given by

$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}.$$

ELEMENT STIFFNESS MATRIX

The element stiffness matrix, $[S]$, which relates the nodal forces to the nodal displacements, can be obtained from the minimization of strain energy functional U:

$$[S] = \frac{\partial U}{\partial\{D\}} \tag{19}$$

where

$$U = \frac{EI}{2}\int_0^L w''w''\,dx + \frac{k}{2}\int_0^L ww\,dx + \frac{k_1}{2}\int_0^L w'w'\,dx. \tag{20}$$

Substituting for w and its derivatives from eqn (15), $[S]$ can be written as

$$[S] = EI\int_0^L \{N''\}^{\mathrm{T}}\{N''\}\,\mathrm{d}x + k\int_0^L \{N\}^{\mathrm{T}}\{N\}\,\mathrm{d}x + k_1\int_0^L \{N'\}^{\mathrm{T}}\{N'\}\,\mathrm{d}x \tag{21}$$

where $\{N\}$ is given by eqn (17) and the first and second derivatives of $\{N\}$, namely $\{N'\}$ and $\{N''\}$, are given by

$$\{N'\} = [G]^{\mathrm{T}}\{\phi'\} \tag{22a}$$

$$\{N''\} = [G]^{\mathrm{T}}\{\phi''\}. \tag{22b}$$

Hence the derivatives of shape functions are directly related to the derivatives of ϕ. The latter derivatives are explicitly given in Appendix A.

Substituting for $\{N\}$ and its derivatives from eqns (17), (22a) and (22b) into eqn (21) and performing the required integrations yields the stiffness matrix in explicit form:

$$[S] = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ & S_{22} & S_{23} & S_{24} \\ & & S_{33} & S_{34} \\ \text{Symm.} & & & S_{44} \end{pmatrix} \tag{23}$$

where

$$S_{11} = S_{33} = 2EI\left[\alpha(\alpha^2-\beta^2)\left(\frac{\beta rt+\alpha RT}{\Delta}\right)\right] \tag{23a}$$

$$S_{12} = -S_{34} = -2EI\left[\frac{(\alpha^2+\beta^2)}{2} + \alpha^2\beta\left(\frac{r^2T^2-t^2R^2}{\Delta}\right)\right] \tag{23b}$$

$$S_{13} = 2EI\left[\alpha(\beta^2-\alpha^2)\left(\frac{\beta RT+\alpha Tr}{\Delta}\right)\right] \tag{23c}$$

$$S_{14} = -S_{23} = -2EI\left[\alpha(\beta^2-\alpha^2)\left(\frac{Tt}{\Delta}\right)\right] \tag{23d}$$

$$S_{22} = S_{44} = 2EI\left[\alpha\left(\frac{\beta rt-\alpha RT}{\Delta}\right)\right] \tag{23e}$$

$$S_{24} = 2EI\left[\alpha\left(\frac{\alpha Tr-\beta Rt}{\Delta}\right)\right] \tag{23f}$$

in which

$$\Delta = \left(\frac{\beta^2t^2-\alpha^2T^2}{\beta}\right).$$

The remaining elements are given by the symmetry of the stiffness matrix.

NODAL LOAD VECTOR

The nodal load vector, $\{P\}$, corresponding to a loading function, $q(x)$, acting from point x_1 to x_2 of the span L, Fig. 3, is given by

$$\{P\} = \int_{x_1}^{x_2} \{N\}q(x)\,\mathrm{d}x. \tag{24}$$

For a distributed moment $m(x)$ acting from x_1 to x_2

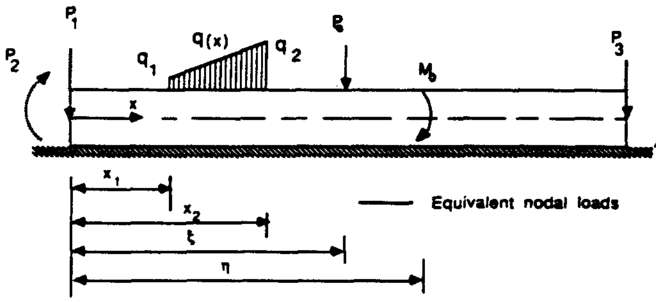

Fig. 3. Equivalent nodal loads P_1–P_4 corresponding to the applied loads.

$$\{P\} = \int_{x_1}^{x_2} \{N'\} m(x)\, \mathrm{d}x. \tag{25}$$

We shall use the above equations to develop the nodal load vector for the most common types of loading as illustrated in Fig. 3.

Concentrated load P_0 *at* $x = \xi$

For this case eqn (22) reduces to

$$\{P\} = P_0\{N\}_{x=\xi}.$$

Substituting for $\{N\}$ from eqn (17) leads to:

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = P_0[G]^T\{\phi\}_{x=\xi}. \tag{26}$$

Concentrated moment M_0 *at* $x = \eta$

Similar to the concentrated load case, the nodal load vector is given by

$$\{P\} = M_0\{N'\}_{x=\eta}$$

or

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = M_0[G]^T\{\phi'\}_{x=\eta}. \tag{27}$$

Trapezoidal load acting from x_1 *to* x_2

The trapezoidal load shown in Fig. 3 has magnitude q_1 at x_1 and magnitude q_2 at x_2. The loading function is given by

$$q(x) = ux + v$$

where

$$u = \frac{q_2 - q_1}{x_2 - x_1}$$

$$v = \frac{q_1 x_2 - q_2 x_1}{x_2 - x_1}.$$

The corresponding nodal load vector is

$$\{P\} = \int_{x_1}^{x_2} \{N\}(ux+v)\,\mathrm{d}x.$$

Substituting for $\{N\}$ from eqn (17),

$$\{P\} = [G]^{\mathrm{T}}\left(u\int_{x_1}^{x_2} x\{\phi\}\,\mathrm{d}x + v\int_{x_1}^{x_2}\{\phi\}\,\mathrm{d}x\right). \tag{28}$$

Carrying out the integration gives

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = [G]^{\mathrm{T}} \begin{pmatrix} W+X+a+b \\ -Y+Z+c-d \\ Y+Z+d+c \\ -W+X+a-b \end{pmatrix} \tag{29}$$

where

$$W = \frac{u}{2\gamma^2}(\cosh \gamma x_1 - \cosh \gamma x_2 + \gamma x_2 \sinh \gamma x_2 - \gamma x_1 \sinh \gamma x_1) \tag{29a}$$

$$X = \frac{u}{2\lambda^2}(\cosh \lambda x_1 - \cosh \lambda x_2 + \lambda x_2 \sinh \lambda x_2 - \lambda x_1 \sinh \lambda x_1) \tag{29b}$$

$$Z = \frac{u}{2\lambda^2}(\sinh \lambda x_1 - \sinh \lambda x_2 + \lambda x_2 \cosh \lambda x_2 - \lambda x_1 \cosh \lambda x_1) \tag{29c}$$

$$Y = \frac{u}{2\gamma^2}(\sinh \gamma x_1 - \sinh \gamma x_2 + \gamma x_2 \cosh \gamma x_2 - \gamma x_1 \cosh \gamma x_1) \tag{29d}$$

$$b = \frac{v}{2\gamma}(\sinh \gamma x_2 - \sinh \gamma x_1) \tag{29e}$$

$$a = \frac{v}{2\lambda}(\sinh \lambda x_2 - \sinh \lambda x_1) \tag{29f}$$

$$d = \frac{v}{2\gamma}(\cosh \gamma x_2 - \cosh \gamma x_1) \tag{29g}$$

$$c = \frac{v}{2\lambda}(\cosh \lambda x_2 - \cosh \lambda x_1) \tag{29h}$$

$$\lambda = \alpha + \beta,$$

$$\gamma = \alpha - \beta.$$

Uniformly distributed moment acting from x_1 *to* x_2

According to eqn (24), the nodal loads for a uniform momemt m_0 acting from point x_1 to x_2 is

$$\{P\} = [G]^{\mathrm{T}} \int_{x_1}^{x_2} \{\phi'\} m_0 \, \mathrm{d}x$$

which, after some manipulations, yields

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = m_0 [G]^{\mathrm{T}} \begin{pmatrix} \phi_1(x_2) - \phi_1(x_1) \\ \phi_2(x_2) - \phi_2(x_1) \\ \phi_3(x_2) - \phi_3(x_1) \\ \phi_4(x_2) - \phi_4(x_1) \end{pmatrix} \tag{30}$$

and where ϕ_1–ϕ_4 are given by eqn (6).

DEFLECTION, BENDING MOMENT AND SHEAR

In order to obtain the displacement, shearing force and bending moment at any point along a beam, we must use the complete displacement function, w, of the beam, consisting of the displacements w_1, caused by nodal displacements $\{D\}$, plus the displacements w_2, caused by the applied loads acting on the same beam with its ends clamped. This can be written as

$$w = w_1 + w_2 \tag{31}$$

where according to eqn (15)

$$w_1 = \{\phi\}^{\mathrm{T}}[G]\{D\} \tag{32a}$$

and

$$w_2 = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4 + w_p \tag{32b}$$

with w_p being a particular integral corresponding to the transverse loading function $q(x)$ and c_1–c_4 being constants of integration. It can be shown that

$$w_p = \frac{1}{k} q(x) + \frac{k_1}{k^2} q''(x) - \frac{1}{k^2}\left(\frac{k_1^2}{k} + kL\right) q^{iv}(x) + \cdots$$

where the primes indicate derivatives of $q(x)$ with respect to x. Equations (31), 32(a) and 32(b) will be used to develop explicit expressions for displacements, shear and bending moment corresponding to some common types of loading.

Linearly varying load

If we restrict our solution to the common case of a linearly varying load, then

$$w_p = \frac{1}{k}(mx + n)$$

where m and n are constants defining the loading function. Therefore,

$$w_2 = \{\phi\}^{\mathrm{T}}\{c\} + \frac{1}{k}(mx + n). \tag{33}$$

The constants $\{c\}$ can be solved using the boundary conditions of a beam clamped at both ends. Insertion of these conditions in eqn (33) leads to

$$[E]\{c\}+\{J\}=0$$

where

$$\{J\}=\frac{1}{k}\begin{pmatrix} n \\ m \\ mL+n \\ m \end{pmatrix}$$

and matrix $[E]$ is the same as in eqn (13). Therefore,

$$\{c\}=-[E]^{-1}\{J\}$$

or

$$\{c\}=-[G]\{J\}. \tag{34}$$

Considering eqns (31)–(34), we can write

$$w=\{\phi\}^{T}[G](\{D\}-\{J\})+\frac{1}{k}(mx+n). \tag{35}$$

Equation (34) gives the complete displacement function which could be used to obtain the displacement at any point along the beam. The shear and bending moment are given by successive differentiation of this equation.

$$V(x)=-EIw'''+k_1w' \tag{36a}$$

$$M(x)=-EIw''. \tag{36b}$$

The shear defined by eqn 36(a) is the vertical shear. The normal shear, acting normal to the deflection line, is

$$V_n(x)=-EIw'''. \tag{37}$$

Equations (35) and (36) can be written in matrix form:

$$V(x)=(-EI\{\phi'''\}^{T}+k_1\{\phi'\}^{T})\{[G](\{D\}-\{J\})\}+\frac{k_1}{k}m \tag{38a}$$

$$M(x)=-EI\{\phi''\}^{T}[G](\{D\}-\{J\}) \tag{38b}$$

$$V_n(x)=-EI\{\phi'''\}^{T}\{[G](\{D\}-\{J\})\}. \tag{39}$$

Once again the derivatives of $\{\phi\}$ are needed in eqns (38) and (39) which can be obtained from Appendix A. It must be mentioned that the vector $\{D\}$ contains the known values of nodal displacements obtained from the analysis.

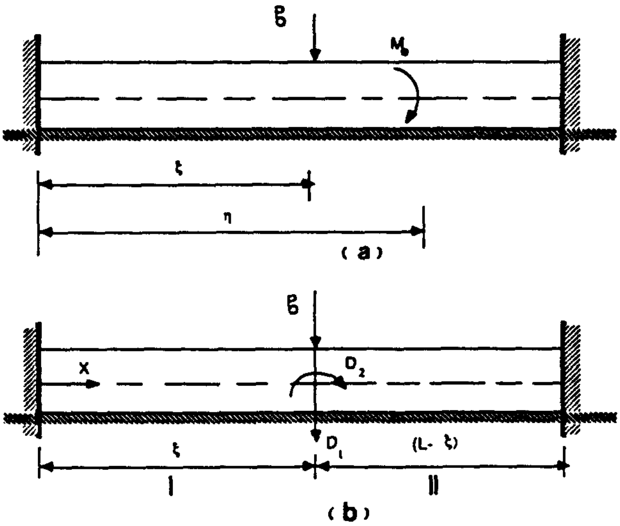

Fig. 4. (a) Beam on two-parameter elastic foundation subjected to concentrated load and moment; (b) beam in (a) idealized as two elements.

Concentrated load P_0 *at* $x = \xi$

In this case the deflected shape, w_1, will have two equations, depending on whether $x < \xi$ or $x \geq \xi$; see Fig. 4(a). To obtain these equations, we treat the beam as an assemblage of two members, I and II, of length ξ and $(L-\xi)$ with two degrees of freedom D_1 and D_2; see Fig. 4(b). We set up the force displacement relationship corresponding to these two degrees of freedom, whence

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} P_0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \frac{P_0}{k_{11}k_{22}-k_{12}k_{21}} \begin{pmatrix} k_{22} \\ -k_{12} \end{pmatrix} \tag{40}$$

where

$$k_{11} = S^{I}_{33}+S^{II}_{11}$$
$$k_{12} = S^{I}_{34}+S^{II}_{12}$$
$$k_{21} = S^{I}_{43}+S^{II}_{21}$$
$$k_{22} = S^{I}_{44}+S^{II}_{22}$$

In the above the superscripts I and II denote members I and II, and the S_{ij} are the stiffness coefficients of each beam element. These can be calculated using the expressions in eqn (21), with the provision that L be replaced by ξ for member I and by $(L-\xi)$ for member II.

Knowing D_1 and D_2 from eqn (40), for $x < \xi$

$$w_2 = D_1N_3+D_2N_4 \tag{41a}$$

while for $x \geq \xi$

$$w_2 = D_1N_1+D_2N_2. \tag{41b}$$

When evaluating N_1 and N_2, we must substitute for x, $(x-\xi)$, or we could use x, provided x is measured from the point of application of the point load.

Concentrated moment M_0 *at* $x = \eta$

The procedure in this case would be similar to the concentrated load case; accordingly,

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \frac{M_0}{k_{11}k_{22} - k_{12}k_{21}} \begin{pmatrix} -k_{21} \\ k_{11} \end{pmatrix} \tag{42}$$

where all the symbols are as defined earlier. In evaluating k_{11}, k_{22}, etc. ξ must be replaced by η.

The expressions for w_2 are the same as those given in eqns (41a) and (41b), except again ξ must be replaced by η. Knowing the expressions for w_2, we could proceed to evaluate the shear force and bending moment using eqns (38a) and (38b), respectively.

NUMERICAL TESTS

To check the accuracy and efficiency of the proposed formulation, several examples were solved. Firstly, a simply supported beam on a two-parameter foundation subjected to a constant line load was analyzed using only one element. The deflections, shearing forces and bending moments were compared with the exact solution given by Hetényi (1961). All the results from the two solutions matched exactly. Subsequently, two beams with more complicated loads were analyzed to demonstrate the efficiency of the formulation. These are described in the following example problems.

Example 1

A free-free beam on a two-parameter elastic foundation, shown in Fig. 5(a), was analyzed exactly by Harr *et al.* (1969) and by Chiwanga and Valsangkar (1988). They took the foundation parameters and beam rigidity such that $k_1 < \sqrt{4kEI}$ and assumed that the foundation does not extend beyond the edges of the beam. The same problem is solved here except that the value of k_1 is changed such that $k_1 > \sqrt{4kEI}$. Two cases are considered in the present analysis. Firstly, it is assumed that the foundation terminates at the beam ends; secondly, the foundation is assumed to be of infinite extent. The results of the first case were compared with those obtained by the foregoing authors. According to the method described here, only one element is required for an exact analysis. Note that for the partial trapezodial load in Fig. 5(a), the particular solution of the governing differential equation is obtained for the loaded and unloaded portions and then the requirements of deflection and slope continuity at their junctures are used to obtain the constants of integration.

The beam loading, dimensions and material properties are shown in Fig. 5(a). The deflected shape, normal shear and bending moment diagrams are shown in Figs 5(b), 5(c) and 5(d), respectively. On the same diagrams results given by Chiwanga and Valsangkar are shown for comparison. We notice that the maximum deflection and the maximum bending moment for the first case are approximately 40% less than their corresponding values but the shear forces do not change much.

If the foundation is assumed to be of infinite extent, then due to the deformations in the part of the foundation beyond the edges of the beam, the shear at the free ends may not generally be zero. The magnitude of this shear force is $(\sqrt{kk_1})w_{f\text{end}}$, where $w_{f\text{end}}$ is the vertical displacement at the free end (Vlasov and Leont'ev, 1966). This is equivalent to having a linear spring of stiffness $\sqrt{kk_1}$ at the free ends. Thus, we can use this simple device of a spring to account for the effect of infinite foundation beyond the beam edges. In the computer program employed in this paper, the proper boundary condition for the free end is automatically accounted for, depending on the specified type foundation. The results for this case are also shown in Figs 5(b)–(d). It can be seen that with the introduction of an infinite foundation the maximum displacement reduces by 50%, but the shear force and bending moment do not change to such a large extent.

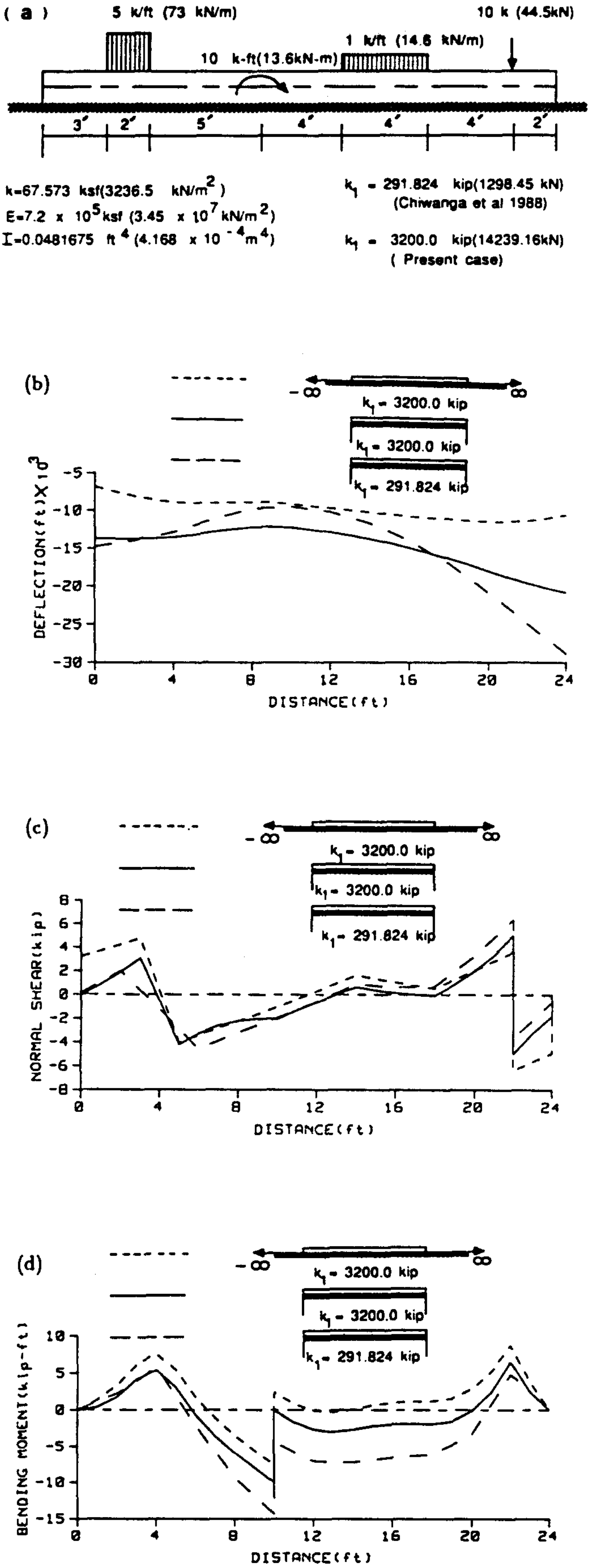

Fig. 5. (a) Free-free beam on two-parameter elastic foundation; (b) deflected shape; (c) normal shear; (d) bending moment.

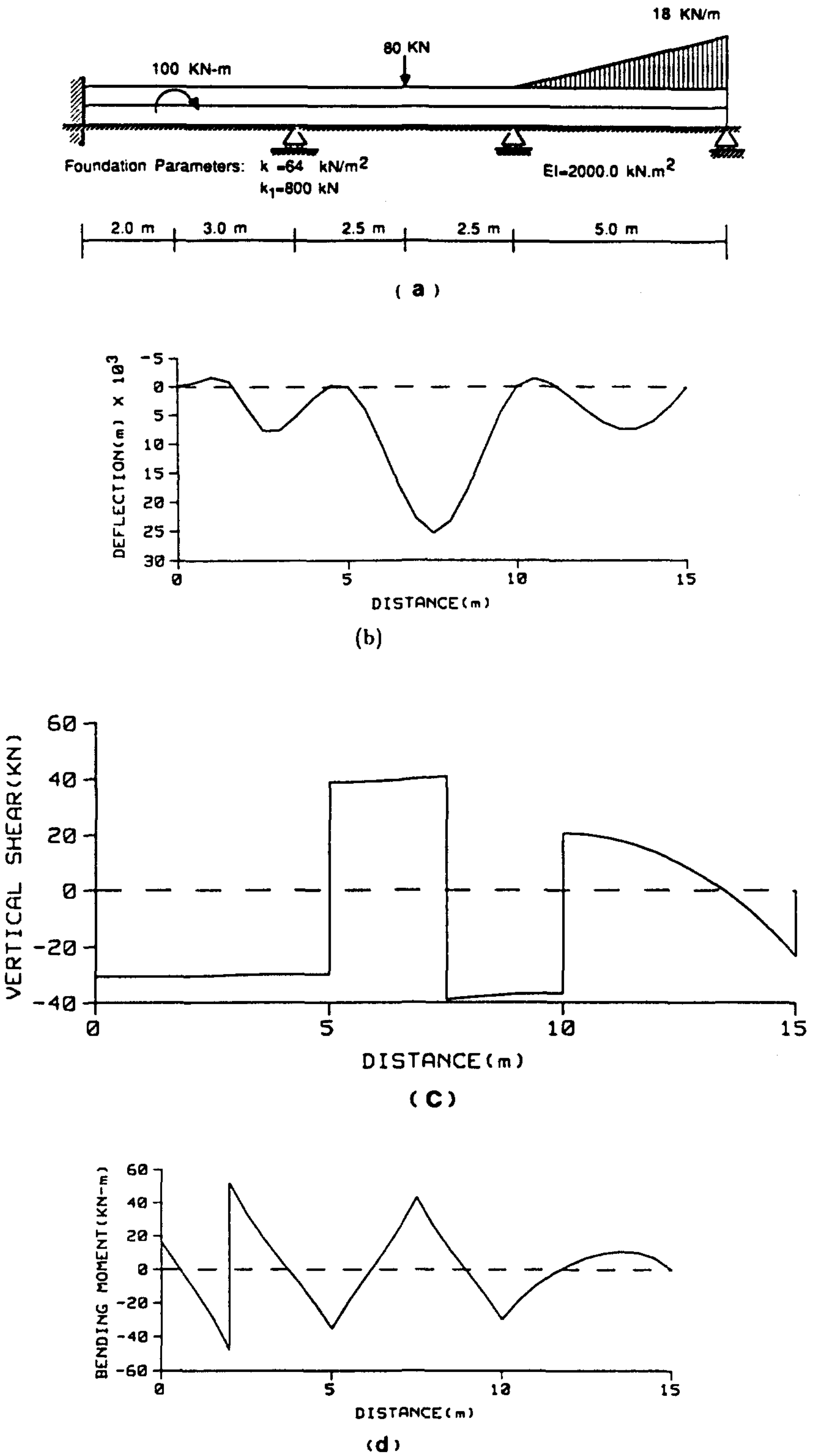

Fig. 6. (a) Continuous beam on two-parameter elastic foundation; (b) deflected shape; (c) vertical shear; (d) bending moment.

Example 2

The continuous beam on two-parameter foundation in Fig. 6(a) is solved using the proposed element. The analysis is performed using three elements and three global degrees of freedom, namely rotations at B, C and D, i.e. D_1, D_2 and D_3. The stiffness matrix of a typical element is calculated using eqn (23).

$$[S]_{\text{element}} = \begin{pmatrix} 498.05 & 626.62 & -341.54 & 514.45 \\ & 2131.05 & -514.45 & 659.46 \\ \text{Symm.} & & 498.05 & -626.62 \\ & & & 2131.05 \end{pmatrix}.$$

Using

$$[S]\{D\} = \{P\},$$

we solve for the joint displacements

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 4262.1 & 659.46 & 0.0 \\ & 4262.1 & 659.46 \\ \text{Symm.} & & 2131.05 \end{pmatrix}^{-1} \begin{pmatrix} 14.09 \\ -27.90 \\ -19.27 \end{pmatrix}$$

which yields

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 4.249 \\ -6.096 \\ -7.157 \end{pmatrix} \times 10^{-3}.$$

Using the displacements $\{D\}$, in conjunction with eqns (37) and (40), the deflected shape, vertical shear and bending moment diagrams of the beam are determined. These are shown in Figs 6(b)–(d), respectively.

SUMMARY AND CONCLUSIONS

An efficient beam on a two-parameter elastic foundation was derived using the exact displacement function obtained from the solution of the governing differential equation. This element complements existing elements of this kind. The stiffness matrix and nodal load vector of the element were derived explicitly, and the detailed equations for the determination of the deflected shape, shearing force, and bending moment were developed. The accuracy and efficiency of the formulation were verified by means of numerical examples. Based on the above, it is concluded that:

(1) Existing beam on two-parameter elastic foundation finite elements cannot be used for certain combinations of foundation parameters and beam rigidity in a convenient and simple manner, to obtain the complete solution of the problem.
(2) The derivation of explicit element stiffness matrix and nodal load vector makes the proposed element efficient and obviates the need for dividing the beam into many elements between the points of loading.
(3) The magnitude of deflections and bending moments are significantly affected by the value of k_1, and by the extent of the foundation if the beam end is free.

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APPENDIX A: ELEMENTS OF THE MATRIX $[G]$ AND THE DERIVATIVES OF $\{\phi\}$

$$G_{11} = 1$$
$$G_{12} = G_{13} = G_{14} = 0$$
$$G_{21} = \frac{\alpha}{\beta}\left(\frac{\beta rt + \alpha RT}{\Delta}\right)$$
$$G_{22} = \left(\frac{t^2}{\Delta}\right)$$
$$G_{23} = -\frac{\alpha}{\beta}\left(\frac{\beta Rt + \alpha Tr}{\Delta}\right)$$
$$G_{24} = \frac{\alpha}{\beta}\left(\frac{Tt}{\Delta}\right)$$
$$G_{31} = -\left(\frac{\beta rt + \alpha RT}{\Delta}\right)$$
$$G_{32} = -\frac{\alpha}{\beta}\left(\frac{T^2}{\Delta}\right)$$
$$G_{33} = \left(\frac{\beta Rt + \alpha Tr}{\Delta}\right)$$
$$G_{34} = -\frac{Tt}{\Delta}$$
$$G_{41} = \alpha\left(\frac{T^2r^2 - R^2t^2}{\Delta}\right)$$
$$G_{42} = \frac{1}{\beta}\left(\frac{\alpha RT - \beta rt}{\Delta}\right)$$
$$G_{43} = \frac{\alpha^2 - \beta^2}{\beta}\left(\frac{Tt}{\Delta}\right)$$
$$G_{44} = \frac{1}{\beta}\left(\frac{\beta Rt - \alpha Tr}{\Delta}\right)$$

and

$$\Delta = \left(\frac{\beta^2 t^2 - \alpha^2 T^2}{\beta}\right).$$

The remaining symbols are the same as in the main text.

Derivatives of $\{\phi\}$:

$$\{\phi'\} = \begin{pmatrix} \beta\phi_2 + \alpha\phi_3 \\ \beta\phi_1 + \alpha\phi_4 \\ \beta\phi_4 + \alpha\phi_1 \\ \beta\phi_3 + \alpha\phi_2 \end{pmatrix}$$

$$\{\phi''\} = \begin{pmatrix} \lambda_2\phi_4 + \lambda_1\phi_1 \\ \lambda_2\phi_3 + \lambda_1\phi_2 \\ \lambda_2\phi_2 + \lambda_1\phi_3 \\ \lambda_2\phi_2 + \lambda_1\phi_4 \end{pmatrix}$$

$$\{\phi'''\} = \begin{pmatrix} \lambda_3\phi_2+\lambda_4\phi_3 \\ \lambda_3\phi_1+\lambda_4\phi_4 \\ \lambda_3\phi_4+\lambda_4\phi_1 \\ \lambda_3\phi_3+\lambda_4\phi_2 \end{pmatrix}$$

where

$$\lambda_1 = \alpha^2+\beta^2$$
$$\lambda_2 = 2\alpha\beta$$
$$\lambda_3 = \beta^3+3\alpha^2\beta$$
$$\lambda_4 = \alpha^3+3\alpha\beta^2.$$

Refer to eqn (6) in the main text for the meaning of ϕ_1–ϕ_4.

APPENDIX B: FORMULATION FOR $k_1 < \sqrt{4kEI}$

Elements of the matrix $[G]$:

$$G_{11} = 1$$
$$G_{12} = G_{13} = G_{14} = 0$$
$$G_{21} = -\frac{\alpha}{\beta_1}\left(\frac{\beta_1 rt+\alpha R_1 T_1}{\Delta_1}\right)$$
$$G_{22} = -\left(\frac{t^2}{\Delta_1}\right)$$
$$G_{23} = \frac{\alpha}{\beta_1}\left(\frac{\beta_1 R_1 t+\alpha T_1 r}{\Delta_1}\right)$$
$$G_{24} = -\frac{\alpha}{\beta_1}\left(\frac{T_1 t}{\Delta_1}\right)$$
$$G_{31} = -\left(\frac{\beta_1 rt+\alpha R_1 T_1}{\Delta_1}\right)$$
$$G_{32} = -\frac{\alpha}{\beta_1}\left(\frac{T_1^2}{\Delta_1}\right)$$
$$G_{33} = \left(\frac{\beta_1 R_1 t+\alpha T_1 r}{\Delta_1}\right)$$
$$G_{34} = -\frac{T_1 t}{\Delta_1}$$
$$G_{41} = \alpha\left(\frac{T_1^2 r^2+R_1^2 t^2}{\Delta_1}\right)$$
$$G_{42} = \frac{-1}{\beta_1}\left(\frac{\alpha R_1 T_1-\beta_1 rt}{\Delta_1}\right)$$
$$G_{43} = -\frac{\alpha^2+\beta_1^2}{\beta_1}\left(\frac{T_1 t}{\Delta_1}\right)$$
$$G_{44} = -\frac{1}{\beta_1}\left(\frac{\beta_1 R_1 t-\alpha T_1 r}{\Delta_1}\right)$$

where

$$R_1 = \sin \beta_1 L$$
$$T_1 = \sin \beta_1 L$$
$$\Delta_1 = \left(\frac{\beta_1^2 t^2-\alpha^2 T_1^2}{\beta_1}\right).$$

The remaining symbols are the same as in the main text.

Derivatives of $\{\phi\}$:

$$\{\phi'\} = \begin{pmatrix} -\beta_1\phi_2+\alpha\phi_3 \\ \beta\phi_1+\alpha\phi_4 \\ -\beta_1\phi_4+\alpha\phi_1 \\ \beta\phi_3+\alpha\phi_2 \end{pmatrix}$$

$$\{\phi''\} = \begin{pmatrix} -\lambda_6\phi_4+\lambda_5\phi_1 \\ \lambda_6\phi_3+\lambda_5\phi_2 \\ -\lambda_6\phi_2+\lambda_5\phi_3 \\ \lambda_6\phi_2+\lambda_5\phi_4 \end{pmatrix}$$

$$\{\phi'''\} = \begin{pmatrix} -\lambda_7\phi_2+\lambda_8\phi_3 \\ \lambda_7\phi_1+\lambda_8\phi_4 \\ -\lambda_7\phi_4+\lambda_8\phi_1 \\ \lambda_7\phi_3+\lambda_8\phi_2 \end{pmatrix}$$

where

$$\lambda_5 = \alpha^2-\beta_1^2$$
$$\lambda_6 = 2\alpha\beta_1$$
$$\lambda_7 = -\beta_1^3+3\alpha^2\beta_1$$
$$\lambda_8 = \alpha^3-3\alpha\beta_1^2.$$

Refer to eqn (8) in the main text for the meaning of ϕ_1–ϕ_4.

Element stiffness matrix:

$$S_{11} = S_{33} = 2EI\left[\alpha(\alpha^2+\beta_1^2)\left(\frac{\beta_1 rt+\alpha R_1 T_1}{\Delta_1}\right)\right]$$

$$S_{12} = -S_{34} = -2EI\left[\frac{(\alpha^2-\beta_1^2)}{2}-\alpha^2\beta_1\left(\frac{r^2T_1^2+t^2R_1^2}{\Delta_1}\right)\right]$$

$$S_{13} = -2EI\left[\alpha(\beta_1^2+\alpha^2)\left(\frac{\beta_1 R_1 t+\alpha T_1 r}{\Delta_1}\right)\right]$$

$$S_{14} = -S_{23} = 2EI\left[\alpha(\beta_1^2+\alpha^2)\left(\frac{T_1 t}{\Delta_1}\right)\right]$$

$$S_{22} = S_{44} = 2EI\left[\alpha\left(\frac{\beta_1 rt-\alpha R_1 T_1}{\Delta_1}\right)\right]$$

$$S_{24} = 2EI\left[\alpha\left(\frac{\alpha T_1 r-\beta_1 R_1 t}{\Delta_1}\right)\right].$$

Formulae of nodal load vectors for all cases, except trapezodial loading, are the same as those given in the main text. The nodal load vector for trapezoidal loading can be evaluated by integrating eqn (28). It is to be noted that for evaluating nodal load vectors, the appropriate $[G]$ and $\{\phi\}$ matrices must be used.